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SOME CONDITIONS FOR P-VALENT STRONGLY STARLIKE FUNCTIONS WITH FIXED SECOND COEFFICIENTS

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Abstract. In the the present paper we investigate some argument properties for analytic functions with fixed second coefficients and positive real part, and we apply our results to certain p-valent functions with fixed second coefficients.

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1. Introduction

Let $\mathbb{U} = \{z : |z| < 1\}$ be the open unit disc of the complex plane $\mathbb{C}$ and let $\mathcal{A}_p$ denote the class of analytic and $p-$valent functions in $\mathbb{U}$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}) \tag{1.1}$$

Also let $\mathcal{A}_1 := \mathcal{A}$. For $0 \leq \alpha < p$ and $p \in \mathbb{N}$, we say that $f(z) \in \mathcal{A}_p$ is in the class $S_p^*(\alpha)$ of p-valent starlike functions of order α if it satisfies the following inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{1.2}$$

Also For $0 \leq \alpha < p$ and $p \in \mathbb{N}$, we say that $f(z) \in \mathcal{A}_p$ is in the class $K_p(\alpha)$ of p-valent convex functions of order α if it satisfies the following inequality

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}). \tag{1.3}$$

The classes $S_p^*(\alpha)$ and $K_p(\alpha)$ were introduced and studied by Owa [13] and Aouf [3]. We note that $S_p^*(0) = S_p^*$ and $K_p(0) = K_p$ are, respectively, the classes of p-valent starlike functions and p-valent convex functions. A function $f(z) \in \mathcal{A}_p$ is said to be $p-$valently $\gamma-$convex of order α if it satisfies the inequality

$$\Re\left\{(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma(1 + \frac{zf''(z)}{f'(z)})\right\} > \alpha \tag{1.4}$$

for some $\gamma(\gamma \geq 0), \alpha(0 \leq \alpha < p, p \in \mathbb{N})$ and for all $z \in \mathbb{U}$. We denote by $J_p(\gamma, \alpha, f(z))$ the subclass of $\mathcal{A}_p$ consisting of all such functions. The class $J_p(\gamma, \alpha, f(z))$ was introduced and studied by Owa [14, with $n = 1$]. We note that $J_p(0, \alpha, f(z) = S_p^*(\alpha)$ and $J_p(1, \alpha, f(z) = K_p(\alpha)$. Also the class

$J_p(\gamma, 0, f(z)) = J_p(\gamma, f(z))$ was introduced and studied by Owa and Ren [15] and Ren and Owa [16]. A function $f(z) \in \mathcal{A}_p$ is said to be p-valently strongly starlike of order $\gamma(0 < \gamma \leq 1)$ if it satisfies the following inequality

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \gamma\frac{\pi}{2} \quad (z \in \mathbb{U}). \tag{1.5}$$

We denote the class of all p-valently strongly starlike of order γ by $S_p(\gamma)$. We note that $S_p(1) = S_p^*$ and $S_1(1) = S^*$. The class $S_p(\gamma)$ was introduced and studied by Kwon and Owa [5] and Liu [8]. Also a function $f(z) \in \mathcal{A}_p$ is said to be p-valently strongly convex of order $\gamma(0 < \gamma \leq 1)$ if it satisfies the following inequality

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \gamma\frac{\pi}{2} \quad (z \in \mathbb{U}). \tag{1.6}$$

We denote the class of all p-valently strongly convex of order γ by $\tilde{K}_p(\gamma)$. We note that $\tilde{K}_p(1) = K_p$ and $\tilde{K}_1(1) = K$. The class $\tilde{K}_p(\gamma)$ was introduced and studied by Muhammad [9].

Let $f(z) \in \mathcal{A}_p$ with $f(z)$, $f'(z)$ and $1+\frac{zf''(z)}{f'(z)} \neq 0$ in $0 < |z| < 1$. Suppose that

$$\mathrm{Re}\left\{\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} > \alpha \quad (0 \leq \gamma < 1;\ p \in \mathbb{N};\ z \in \mathbb{U}), \tag{1.7}$$

where the powers appearing in (1.7) are meant as the principal values. We say that $f(z)$ is a p-valent γ-starlike function of order $\alpha(0 \leq \alpha < 1)$ and we denote the class of such functions by $M_p^{\gamma}(\alpha)$. We note that $M_p^0(0) = S_p^*$ and $M_p^1(0) = K_p$. Also the classes $M_p^{\gamma}(0) = M^{\gamma}$ and $M_1^{\gamma}(\alpha) = M^{\gamma}(\alpha)$ are introduced and studied by Lawandowski et al. [7]. They proved that $M^{\gamma} \subset S^*$ for all real γ.

Remark 1.1. *The condition (1.7) is equivalent to the following condition*

$$\left|(1-\gamma)\arg\left(\frac{zf'(z)}{f(z)}\right) + \gamma\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \ (0 \leq \gamma \leq 1;\ 0 < \alpha \leq 1;\ z \in \mathbb{U}).$$

We note that $M_p^0(\alpha) = S_p(\alpha)$ and $M_p^1(\alpha) = \tilde{K}_p(\alpha)$.

Let $A(p, \beta)$ consists of analytic functions $f(z) \in \mathcal{A}_p$ of the form

$$f(z) = z^p + \beta z^{p+1} + a_{p+2}z^{p+2} + ..., \tag{1.8}$$

where $p \in \mathbb{N}$ and the second coefficient $\beta \in \mathbb{C}$ ($\mathbb{C}$ the complex plane) is fixed constant. Several authors have investigated functions with fixed second coefficient when $p = 1$, we see for example, by Ali et al. [1, 2], Nagpal and Ravichandran [10], Lashin and El-Emam [6] and Kwon [4].

Now, for fixed $\beta \in \mathbb{C}$ and $p \in \mathbb{N}$, let $P(p, \beta)$ consists of functions $q(z)$ of the form

$$q(z) = p + \beta z + p_2 z^2 + ..., \tag{1.9}$$

where the second coefficient $\beta \in \mathbb{C}$ is fixed constant.

In this paper, we investigate some argument properties for analytic functions with fixed second coefficient and positive real part and apply our results to certain subclasses of p-valent functions with fixed second coefficients.

We need the following lemmas for functions with fixed initial coefficient.

Lemma 1.1. [1] *Let* $z_0 = r_0e^{-i\theta_0}(r_0 < 1)$ *and* $g(z) = g_nz^n + g_{n+1}z^{n+1} + g_{n+2}z^{n+2} + ...$ *be continuous in* $\mathbb{U}_{r_0}$, *analytic in* $\mathbb{U}_{r_0} \cup \{z_0\}$ *with* $g(z) \neq 0$ *and* $n \geq 1$. *If*

$$|g(z_0)| = \max_{|z|\leq|z_0|} |g(z)|,$$

then $\frac{z_0g'(z_0)}{g(z_0)} = m$ *and* $\operatorname{Re}\left(1 + \frac{z_0g''(z_0)}{g'(z_0)}\right) \geq m$, *where*

$$m \geq n + \frac{|g(z_0)| - |g_n|\, r_0^n}{|g(z_0)| + |g_n|\, r_0^n}. \tag{1.10}$$

Remark 1.2. *we note that the inequality (1.10) implies that*

$$m \geq n + \frac{|g(z_0)| - |g_n|}{|g(z_0)| + |g_n|},$$

since $r_0 < 1$.

Lemma 1.2. *Let* $q(z) \in P(p, \beta)$ *be analytic in* $\mathbb{U}$ *and* $q(z) \neq 0$ *in* $\mathbb{U}$. *Suppose that there exists a point* $z_0 \in \mathbb{U}$ *such that* $\operatorname{Re}\{q(z)\} > 0$ *for* $|z| < |z_0|$ *and* $\operatorname{Re}\{q(z_0)\} = 0$ *and* $q(z_0) \neq 0$. *Then*

$$\frac{z_0q'(z_0)}{q(z_0)} = ik,$$

where $k \in \mathbb{R}$ *and* $k \geq \frac{4p}{2p+|\beta|}$.

Proof. Let us put

$$\varphi(z) = \frac{p - q(z)}{p + q(z)} \quad (z \in \mathbb{U}).$$

Then $\varphi(0) = 0,\ |\varphi(z)| < 1$ for $|z| < |z_0|$ and $|\varphi(z_0)| = 1$. We note that

$$\varphi(z) = -\frac{\beta}{2p}z + \left(\frac{\beta^2}{4p^2} - \frac{p_2}{2p}\right) z^2 + \cdots .$$

From Lemma 1.1, we have

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)} \geq \frac{4p}{2p + |\beta|}.$$

Hence

$$\frac{z_0\varphi'(z_0)}{\varphi(z_0)} = \frac{-2pz_0q'(z_0)}{p^2 - (q(z_0))^2} = \frac{-2pz_0q'(z_0)}{p^2 + |q(z_0)|^2} \geq \frac{4p}{2p + |\beta|}.$$

This inequality implies $z_0q'(z_0)$ is a negative real number which satisfies

$$z_0q'(z_0) \leq \frac{-2}{2p + |\beta|}\left(p^2 + |q(z_0)|^2\right).$$

Now, we put $q(z_0) = ia$. For the case $a > 0$, we have

$$\operatorname{Im}\left\{\frac{z_0 q'(z_0)}{q(z_0)}\right\} = \operatorname{Im}\left\{\frac{-iz_0 q'(z_0)}{|q(z_0)|}\right\} = -\frac{z_0 q'(z_0)}{|q(z_0)|}$$

$$\geq \frac{2}{2p+|\beta|}\frac{p^2+|q(z_0)|^2}{|q(z_0)|} = \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right). \tag{1.11}$$

For the case, $a < 0$, we have

$$\operatorname{Im}\left\{\frac{z_0 q'(z_0)}{q(z_0)}\right\} = \operatorname{Im}\left\{\frac{iz_0 q'(z_0)}{|q(z_0)|}\right\} = \frac{z_0 q'(z_0)}{|q(z_0)|}$$

$$\leq \frac{-2}{2p+|\beta|}\frac{p^2+|q(z_0)|^2}{|q(z_0)|} = \frac{-2}{2p+|\beta|}\left(\frac{p^2+a^2}{|a|}\right). \tag{1.12}$$

From (1.11) and (1.12), we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where $k \in \mathbb{R}$ and

$$k \geq \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right).$$

This completes the proof of Lemma 1.2. □

Lemma 1.3. *Let $q(z) \in P(p, \beta)$ be analytic in $\mathbb{U}$, and $q(z) \neq 0$ $(z \in \mathbb{U})$. If there exists a point $z_0 \in \mathbb{U}$, such that*

$$|\arg(q(z))| < \frac{\pi}{2}\alpha \quad \textit{for } |z| < |z_0|$$

and

$$|\arg(q(z_0))| = \frac{\pi}{2}\alpha \quad (\alpha > 0),$$

then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha,$$

where

$$k \geq \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \ \textit{when } \arg(q(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \leq -\frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \ \textit{when } \arg(q(z_0)) = -\frac{\pi}{2}\alpha \tag{1.13}$$

with $\{q(z_0)\}^{\frac{1}{\alpha}} = \pm ia$.

Proof. We put

$$g(z) = \{q(z)\}^{\frac{1}{\alpha}}.$$

Then $\operatorname{Re}(g(z)) > 0$ for $|z| < |z_0|$ and $\operatorname{Re}(g(z_0)) = 0$. Let us put

$$g(z_0) = \pm ia \ (a > 0).$$

Applying Lemma 1.2, we have

$$\frac{z_0 g'(z_0)}{g(z_0)} = \frac{z_0 q'(z_0)}{\alpha q(z_0)} = ik,$$

where $k \in \mathbb{R}$ with

$$k \geq \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \text{ for } (g(z_0)) = ia,$$
$$k \leq -\frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \text{ for } (g(z_0)) = -ia.$$

This completes the proof of Lemma 1.3. □

Remark 1.3. (1) *Putting $p = 1$ and $\beta = 2$ in Lemma 1.3, we obtain the result obtained by Nunokawa* [11, Lemma 2].
(2) *Putting $p = 1$ in Lemma 1.2 and Lemma 1.3, respectively, we obtain the results obtained by Kwon* [4, Lemma 1 and Lemma 2, respectively]*.*

2. Main Results

Theorem 2.1. *Let $\gamma > 0$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$ and $q(z) \in P(p,\beta)$ satisfy*

$$\left|\arg\left\{q(z) + \gamma\frac{zq'(z)}{q(z)}\right\}\right| < \frac{\pi}{2}\delta \ (z \in \mathbb{U}), \tag{2.1}$$

then

$$|\arg(q(z))| < \frac{\pi}{2}\eta \qquad (z \in \mathbb{U};\ 0 < \eta < 1), \tag{2.2}$$

where

$$\delta = \eta + \frac{2}{\pi}\arctan\left(\frac{4p^2\eta\gamma\sin\ (\pi(1-\eta)/2)}{(2p+|\beta|)(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)} + 4p^2\gamma\eta\cos\ (\pi(1-\eta)/2)}\right). \tag{2.3}$$

Proof. Suppose that there exists a point $z_0 \in \mathbb{U}$, such that

$$|\arg(q(z))| < \frac{\pi}{2}\eta \ \text{ for } |z| < |z_0| \text{ and } \quad |\arg(q(z_0))| = \frac{\pi}{2}\eta \ \ (0 < \eta \leq 1),$$

By using Lemma 1.3, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\eta k,$$

where

$$k \geq \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \text{ for } \arg(q(z_0)) = \frac{\pi}{2}\eta,$$
$$k \leq -\frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right), \text{ for } \arg(q(z_0)) = -\frac{\pi}{2}\eta$$

with $\{q(z_0)\}^{\frac{1}{\eta}} = \pm ia$. For the case $q(z_0) = (ia)^{\eta}$ $(a > 0)$, we have

$$\begin{aligned}\arg\left\{q(z_0) + \gamma\frac{zq'(z_0)}{q(z_0)}\right\} &= \arg\left(q(z_0)\right) + \arg\left\{1 + \gamma\frac{zq'(z_0)}{(q(z_0))^2}\right\} \\ &= \frac{\pi}{2}\eta + \arg\left(1 + \gamma\frac{i\eta k}{(ia)^{\eta}}\right) \\ &= \frac{\pi}{2}\eta + \arctan\left(\frac{\gamma\frac{\eta k}{a^{\eta}}\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1+\gamma\frac{\eta k}{a^{\eta}}\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right).\end{aligned}$$

Since

$$\frac{\eta k}{a^{\eta}} \geq \frac{2\eta}{2p+|\beta|}\left(a^{1-\eta} + p^2 a^{-1-\eta}\right).$$

Now, we define a function $g_p : (0,\infty) \to \mathbb{R}$ by

$$g_p(a) = \frac{2\eta}{2p+|\beta|}\left(a^{1-\eta} + p^2 a^{-1-\eta}\right) \quad (p \in \mathbb{N};\ a > 0).$$

Then $g_p(a)$ takes the minimum value at $a = p\left((1+\eta)/(1-\eta)\right)^{1/2}$ because

$$g_p'(a) = \frac{2\eta(1-\eta)}{(2p+|\beta|)a^{\eta+2}}\left(a^2 - p^2\frac{1+\eta}{1-\eta}\right).$$

Thus we have

$$\begin{aligned}&\arg\left\{q(z_0) + \gamma\frac{zq'(z_0)}{q(z_0)}\right\} \\ &\geq \frac{\pi}{2}\eta + \arctan\left(\frac{\frac{2\eta\gamma}{2p+|\beta|}\left[\left(\frac{p^2+p^2\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)} + p^2\left(\frac{1-\eta}{p^2+p^2\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\sin\left(\frac{\pi(1-\eta)}{2}\right)}{1+\frac{2\eta\gamma}{2p+|\beta|}\left[\left(\frac{p^2+p^2\eta}{1-\eta}\right)^{\frac{1}{2}(1-\eta)} + p^2\left(\frac{1-\eta}{p^2+p^2\eta}\right)^{\frac{1}{2}(1+\eta)}\right]\cos\left(\frac{\pi(1-\eta)}{2}\right)}\right) \\ &= \frac{\pi}{2}\eta + \arctan\left(\frac{4p^2\eta\gamma\sin\left(\pi(1-\eta)/2\right)}{(2p+|\beta|)(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)} + 4p^2\gamma\eta\cos\left(\pi(1-\eta)/2\right)}\right) \\ &= \frac{\pi}{2}\delta,\end{aligned}$$

which contradicts the condition (2.3). If $q(z_0) = (-ia)^{\eta}$ $(a > 0)$, applying same method as above, we also have

$$\begin{aligned}&\arg\left\{q(z_0) + \gamma\frac{zq'(z_0)}{q(z_0)}\right\} \\ &\leq -\frac{\pi}{2}\eta - \arctan\left(\frac{4p^2\eta\gamma\sin\left(\pi(1-\eta)/2\right)}{(2p+|\beta|)(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)} + 4p^2\gamma\eta\cos\left(\pi(1-\eta)/2\right)}\right) \\ &= -\frac{\pi}{2}\delta,\end{aligned}$$

which contradicts the condition (2.3). This implies that there is no point $z_0 \in \mathbb{U}$, such that

$$|\arg\left(q(z_0)\right)| = \frac{\pi}{2}\eta.$$

This completes the proof of Theorem 2.1. □

Remark 2.1. (1) *Putting* $p=1$ *and* $\beta=2$ *in Theorem 2.1,we obtain the result obtained by Nunokawa et al.* [12, Theorem 3].

(2) *Putting* $p=1$ *and replacing* γ *by* $(1-\gamma)(0\leq\gamma\leq 1)$ *in Theorem 2.1, we obtain the result obtained by Kwon* [4, Theorem 4].

Putting $\beta=2p(p\in\mathbb{N})$ and $q(z)=\frac{zf'(z)}{f(z)},\ f(z)\in\mathcal{A}_p$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let* $\gamma>0,\ p\in\mathbb{N}$ *and* $f(z)\in\mathcal{A}_p$ *satisfy*

$$\left|\arg\left\{(1-\gamma)\left(\frac{zf'(z)}{f(z)}\right)+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right|<\frac{\pi}{2}\delta\quad(z\in\mathbb{U}). \tag{2.4}$$

Then $f(z)$ *is* p*-valently strongly starlike of order* $\eta(0<\eta<1)$, *where*

$$\delta=\eta+\frac{2}{\pi}\arctan\left(\frac{p\eta\gamma\sin\ (\pi(1-\eta)/2)}{(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)}+p\gamma\eta\cos\ (\pi(1-\eta)/2)}\right).$$

Taking $\gamma=1$ in Corollary 2.1, we have the following corollary.

Corollary 2.2. *Let* $f(z)\in\mathcal{A}_p$ *satisfy*

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\pi}{2}\delta\quad(z\in\mathbb{U}). \tag{2.5}$$

Then $f(z)$ *is* p*-valently strongly starlike of order* $\eta(0<\eta<1)$, *where*

$$\delta=\eta+\frac{2}{\pi}\arctan\left(\frac{p\eta\sin\ (\pi(1-\eta)/2)}{(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)}+p\eta\cos\ (\pi(1-\eta)/2)}\right).$$

Theorem 2.2. *Let* $\gamma>0,\ \beta\in\mathbb{C},\ p\in\mathbb{N}$ *and* $q(z)\in P(p,\beta)$ *satisfy*

$$\left|\arg(q(z))+\gamma\arg\left(1+\frac{zq'(z)}{(q(z))^2}\right)\right|<\frac{\pi}{2}\delta\quad(z\in\mathbb{U}). \tag{2.6}$$

Then

$$|\arg(q(z))|<\frac{\pi}{2}\eta\qquad(z\in\mathbb{U};\ 0<\eta<1), \tag{2.7}$$

where

$$\delta=\eta+\frac{2\gamma}{\pi}\arctan\left(\frac{4p^2\eta\sin\ (\pi(1-\eta)/2)}{(2p+|\beta|)\,(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)}+4p^2\eta\cos\ (\pi(1-\eta)/2)}\right). \tag{2.8}$$

Proof. If there exists a point $z_0\in\mathbb{U}$, such that

$$|\arg(q(z))|<\frac{\pi}{2}\eta\ \text{ for } |z|<|z_0|\ \text{ and }\ |\arg(p(z_0))|=\frac{\pi}{2}\eta\ (0<\eta<1),$$

then, by using Lemma 1.3, we have

$$\frac{z_0q'(z_0)}{q(z_0)}=i\eta k.$$

If $\arg(q(z_0))=\frac{\pi}{2}\eta$, then we have $\{q(z_0)\}^{\frac{1}{\eta}}=ia(a>0)$. Therefore, we see that

$$q(z_0)\left(1+\frac{zq'(z_0)}{(q(z_0))^2}\right)^{\gamma}=a^{\eta}e^{i\frac{\pi}{2}\eta}\left(1+\frac{\eta k}{a^{\eta}}e^{i\frac{\pi}{2}(1-\eta)}\right)^{\gamma},$$

where

$$k \geq \frac{2}{2p+|\beta|}\left(\frac{p^2+a^2}{a}\right).$$

Hence we have

$$\frac{\eta k}{a^\eta} \geq \frac{2\eta}{2p+|\beta|}\left(a^{1-\eta}+p^2a^{-1-\eta}\right).$$

Now, we define a function $g_p : (0,\infty) \to \mathbb{R}$ by

$$g_p(a) = \frac{2\eta}{2p+|\beta|}\left(a^{1-\eta}+p^2a^{-1-\eta}\right) \quad (p \in \mathbb{N};\ a>0).$$

By applying the same method and technique as in our proof of Theorem 2.1, $g_p(a)$ takes the minimum value at $a = p\sqrt{(1+\eta)/(1-\eta)}$. Thus we have

$$\begin{aligned}
&\arg(q(z_0)) + \gamma \arg\left(1+\frac{zq'(z_0)}{(q(z_0))^2}\right)\\
&= \frac{\pi}{2}\eta + \gamma\arg\left(1+\frac{\eta k}{a^\eta}e^{i\frac{\pi}{2}(1-\eta)}\right)\\
&\geq \frac{\pi}{2}\eta + \gamma\arctan\left(\frac{4p\eta\sin\ (\pi(1-\eta)/2)}{(2p+|\beta|)\,(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)}+4p\eta\cos\ (\pi(1-\eta)/2)}\right)\\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

which contradicts the condition (2.8). If $\arg(q(z_0)) = -\frac{\pi}{2}\eta$, applying the same method as above, we have

$$\arg(q(z_0)) + \gamma\arg\left(1+\frac{zq'(z_0)}{(q(z_0))^2}\right) \leq -\frac{\pi}{2}\delta,$$

which contradicts (2.8). This completes the proof of Theorem 2.2. □

Remark 2.2. (1) *Putting $p=1$ and $\beta=2$ in Theorem 2.2, we obtain the result obtained by Nunokawa et al.* [12, Theorem 1].
(2) *Putting $p=1$ in in Theorem 2.2, we obtain the results obtained by Kwon* [4, Theorem 6].

Putting $q(z) = zf'(z)/f(z),\ f(z) \in \mathcal{A}_p$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $\gamma>0,\ p\in\mathbb{N},\ \beta\in\mathbb{C}$ and $f(z)\in\mathcal{A}_p$ satisfy*

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \frac{\pi}{2}\delta\ (z\in\mathbb{U}). \tag{2.9}$$

Then $f(z)$ is p-valently strongly starlike of order $\eta(0<\eta<1)$, where δ is given by (2.8).

Putting $\beta = 2p(p\in\mathbb{N})$ in Corollary 2.3, we have the following corollary.

Corollary 2.4. *Let $\gamma>0,\ p\in\mathbb{N}$ and $f(z)\in\mathcal{A}_p$ satisfy (2.9). Then $f(z)$ is p-valently strongly starlike of order $\eta(0<\eta<1)$, where*

$$\delta = \eta + \frac{2\gamma}{\pi}\arctan\left(\frac{p\eta\sin\ (\pi(1-\eta)/2)}{(1-\eta)^{\frac{1}{2}(1-\eta)}(p^2+p^2\eta)^{\frac{1}{2}(1+\eta)}+p\eta\cos\ (\pi(1-\eta)/2)}\right).$$

Remark 2.3. *Putting $p = 1$in Corollary 2.4,we obtain the result obtained by Nunokawa et al.* [12, Theorem 1].

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